

The absolute gradings in embedded contact homology and Seiberg-Witten Floer cohomology

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Abstract

Let Y be a closed connected contact 3-manifold. In [12], Taubes defines an isomorphism between the embedded contact homology (ECH) of Y and its Seiberg-Witten Floer cohomology. Both the ECH of Y and the Seiberg-Witten Floer cohomology of Y admit absolute gradings by homotopy classes of two plane fields. We show that Taubes' isomorphism preserves these gradings.

1 Introduction

Let Y be a closed connected oriented 3-manifold. A *contact form* on Y is a 1-form λ such that $\lambda \wedge d\lambda > 0$. A contact form determines the *Reeb vector field*, R , by the equations $d\lambda(R, \cdot) = 0$, $\lambda(R) = 1$, and an oriented 2-plane field $\xi := \text{Ker}(\lambda)$, called the *contact structure* for λ . A *Reeb orbit* is a map $\gamma : \mathbb{R}/T\mathbb{R}$ for some $T > 0$ such that $\gamma'(t) = R(\gamma(t))$. If γ is a Reeb orbit, then γ is called *nondegenerate* if, for some y on the image of γ , the linearized flow along γ restricted to ξ_y does not have 1 as an eigenvalue. If γ is nondegenerate and the eigenvalues of the linearized flow are real, then γ is said to be *hyperbolic*; otherwise, γ is called *elliptic*. This does not depend on the choice of y . A contact form λ is called *nondegenerate* if all of its Reeb orbits are nondegenerate.

If λ is nondegenerate and $\Gamma \in H_1(Y)$, then the *embedded contact homology* (with $\mathbb{Z}/2$ coefficients) $ECH(Y, \lambda, \Gamma)$ of Y is defined. This is the homology of a chain complex freely generated over $\mathbb{Z}/2$ by finite sets $\{(\alpha_i, m_i)\}$, such that each α_i is a Reeb orbit, m_i is equal to 1 if α_i is hyperbolic, and

$$\sum_i m_i [\alpha_i] = \Gamma \in H_1(Y). \quad (1.1)$$

The $\{(\alpha_i, m_i)\}$ are called *orbit sets*. The chain complex differential counts certain J -holomorphic curves in the symplectization $\mathbb{R} \times Y$ of Y for generic “admissible” J . Embedded contact homology can also be defined with \mathbb{Z} coefficients, see [9, §9], but in this paper, we always assume that we are using $\mathbb{Z}/2$ coefficients. The definition of ECH will be reviewed in §2.1.

The contact form λ and the almost complex structure J determine a metric on Y , characterized by the equations $*d\lambda = 2\lambda$ and $|\lambda| = 1$. Recall that a *spin^c structure* on a Riemannian 3-manifold Y is a unitary, rank 2 complex vector bundle \mathbb{S} , together with a *Clifford multiplication* $\rho : TY \rightarrow \text{Hom}(\mathbb{S}, \mathbb{S})$. Clifford multiplication is defined by the property that given any $y \in Y$, there exists an oriented orthonormal basis $\{e_1, e_2, e_3\}$ for $T_y Y$ and an orthonormal basis for \mathbb{S}_y such that

the linear transformations $\rho(e_1), \rho(e_2)$, and $\rho(e_3)$ are given in these coordinates by the matrices

$$\rho(e_1) = \sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \rho(e_2) = \sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho(e_3) = \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \quad (1.2)$$

If Y is equipped with a spin^c structure \mathfrak{s} , then the *Seiberg-Witten Floer cohomology* (with $\mathbb{Z}/2$ coefficients) $\widehat{HM}^*(Y, \mathfrak{s})$ of Y is defined. Roughly speaking (the details will be reviewed in §3.3), this is the homology of a chain complex generated over $\mathbb{Z}/2$ by gauge equivalence classes of solutions to the *three-dimensional Seiberg-Witten equations* with respect to a differential that counts certain solutions to the *four-dimensional Seiberg-Witten equations* on $\mathbb{R} \times Y$. Seiberg-Witten Floer cohomology can also be defined with \mathbb{Z} coefficients (see [11]) but in this paper we again always assume that we are working with $\mathbb{Z}/2$ coefficients.

A fundamental result of Taubes asserts that the embedded contact homology and Seiberg-Witten Floer cohomology of Y are isomorphic. Specifically, Taubes shows ([13, Thm. 1]) that there is a canonical isomorphism of $\mathbb{Z}/2$ modules

$$T : ECH_*(Y, \lambda, \Gamma) \simeq \widehat{HM}^{-*}(Y, \mathfrak{s}_\xi + \text{PD}(\Gamma)). \quad (1.3)$$

Here, \mathfrak{s}_ξ is a certain spin^c structure determined by ξ , as reviewed in [6, §8], and $\text{PD}(\Gamma)$ denotes the Poincaré dual of Γ .

The $\mathbb{Z}/2$ -modules $ECH_*(Y, \lambda, \Gamma)$ and $\widehat{HM}^{-*}(Y, \mathfrak{s}_\xi + \text{PD}(\Gamma))$ admit a relative \mathbb{Z}/p grading, where p denotes the divisibility of $c_1(\xi) + 2\text{PD}(\Gamma)$ in $H^2(Y; \mathbb{Z}) \bmod$ torsion. Taubes shows ([15, Thm. 1.1]) that the map T reverses the sign of this grading. Denote by $ECH(Y, \lambda)$ the direct sum of $ECH(Y, \lambda, \Gamma)$ over all Γ and by $\widehat{HM}^*(Y)$ the direct sum of $\widehat{HM}^*(Y, \mathfrak{s})$ over all isomorphism classes of spin^c structures on Y . As reviewed in §2.2 and §3.4, $ECH_*(Y, \lambda)$ and $\widehat{HM}^*(Y)$ both admit canonical gradings by homotopy classes of 2-plane fields refining the relative \mathbb{Z}/p grading in each spin^c structure. It is therefore natural to conjecture (see [6, Thm. 7.4]) that T preserves this structure as well. The main theorem of this paper asserts that this does in fact hold.

Specifically, let j be a homotopy class of oriented 2-plane fields on Y . We show

Theorem 1.1. *The map T induces an isomorphism*

$$T_a : ECH_j(Y, \lambda) \simeq \widehat{HM}^j(Y). \quad (1.4)$$

2 Embedded contact homology

We begin by reviewing the aspects of embedded contact homology that are relevant to the proof of Theorem 1.1.

2.1 Definition of embedded contact homology

We will first review the definition of embedded contact homology (for more details, see [3]). Let J be an “admissible” \mathbb{R} -invariant almost complex structure on $\mathbb{R} \times Y$.

This means that J sends the two-plane field ξ to itself, rotating it positively with respect to $d\lambda$, and satisfies $J(\partial_s) = R$, where s denotes the \mathbb{R} coordinate on $\mathbb{R} \times Y$. Define $ECC(Y, \lambda, \Gamma, J)$ to be the chain complex generated over $\mathbb{Z}/2$ by finite sets $\alpha = \{(\alpha_i, m_i)\}$ such that each α_i is a Reeb orbit, $m_i = 1$ if α_i is hyperbolic, and

$$\sum_i m_i [\alpha_i] = \Gamma \in H_1(Y).$$

The chain complex differential ∂_{ECH} counts certain J -holomorphic curves in $\mathbb{R} \times Y$: if α and β are two chain complex generators, then the coefficient $\langle \partial\alpha, \beta \rangle \in \mathbb{Z}/2$ is a count of J -holomorphic curves in $\mathbb{R} \times Y$, modulo translation in the \mathbb{R} coordinate, that are asymptotic as currents to $\mathbb{R} \times \alpha$ as $s \rightarrow \infty$ and to $\mathbb{R} \times \beta$ as $s \rightarrow -\infty$ and which have *ECH index* 1. The ECH index, a certain function of the relative homology class of the curve, will be reviewed in §2.3. If J is generic, then ∂ is well-defined and $\partial^2 = 0$, see [7] and [9].

Define $ECH(Y, \lambda, \Gamma)$ to be the homology of this chain complex. A priori, this might depend on J , but by the canonical isomorphism (1.3) it does not. The ECH index induces a relative \mathbb{Z}/p grading on $ECH(Y, \lambda, \Gamma)$, as reviewed in §2.3, where p denotes the divisibility of $c_1(\xi) + 2 \text{PD}(\Gamma)$ in $H^2(Y)$ mod torsion.

2.2 The absolute grading for ECH

We now briefly review the definition of the absolute grading for ECH, see [5] for more details.

Recall that a link L in Y is *transversal* if L is transverse to the contact plane field at every point. Let L be a transversal link and orient L so that it intersects the contact plane field positively. A framing of L is equivalent to a homotopy class of symplectic trivializations of $\xi|_L$. Given a transversal link L with framing τ , we can define a homotopy class of 2-plane fields, denoted $P_\tau(L)$. This is explained below.

Let N be a tubular neighborhood of L . On N , choose disjoint tubular neighborhoods N_K for each component K of the link, and choose coordinates $\varphi_K : N_K \xrightarrow{\sim} S^1 \times D^2$ such that φ_K sends K to $S^1 \times \{0\}$ and $d\varphi_K$ sends $\xi|_K$ to $0 \times \mathbb{R}^2$ compatibly with τ ; extend φ_K to a trivialization of the tangent bundle $TN_K \xrightarrow{\sim} \mathbb{R} \times \mathbb{R}^2$ such that the contact plane field is given by $\{0\} \times \mathbb{R}^2$ and the Reeb vector field is given by $(1, 0, 0)$ in these coordinates.

Next, choose a vector field P such that on $S^1 \times \{z \in D^2 \mid |z| > 1/2\}$, the vector field P intersects ξ positively, on $S^1 \times \{z \in D^2 \mid |z| < 1/2\}$ the vector field P intersects ξ negatively, and on $S^1 \times \{z \in D^2 \mid |z| = 1/2\}$, the vector field P is given according to the above trivialization by

$$P(t, e^{i\theta}/2) := (0, e^{-i\theta}). \quad (2.1)$$

A homotopy class of vector fields determines a homotopy class of 2-plane fields. On N , define $P_\tau(L)$ to be the 2-plane field determined by this vector field. On $Y \setminus N$, set $P_\tau(L)$ equal to ξ .

To associate a homotopy class of two-plane fields to an orbit set $\alpha = \{(\alpha_i, m_i)\}$, first choose trivializations $\tau = \{\tau_i\}$ of ξ over each α_i . Next, choose disjoint tubular neighborhoods N_i of the α_i . Finally, in each N_i , choose a braid ζ_i with m_i strands around each α_i (this means that ζ_i is an oriented link in N_i such that the projection

of ζ_i to α_i is a degree m orientation preserving submersion), and define L to be the union of these braids, with the framing induced by τ . Then, $I(\alpha)$ is given by the formula

$$I(\alpha) := P_\tau(L) - \sum_i w_{\tau_i}(\zeta_i) + \mu_\tau(\alpha). \quad (2.2)$$

Here, $w_{\tau_i}(\zeta_i)$ is the writhe of the link ζ_i with respect to τ_i as defined in [5, §2.6], and $\mu_\tau(\alpha)$ is a certain sum of Conley-Zehnder index terms associated to α , see [5, §2.8] for the precise definitions.

It is shown in [5, Lem. 3.7] that $I(\alpha)$ is well-defined. The homotopy class of 2-plane fields $I(\alpha)$ is the canonical grading in ECH.

2.3 Symplectic cobordisms and the ECH index

The proof of Theorem 1.1 also involves the ECH index. We now briefly review this construction for an arbitrary symplectic cobordism.

Let (Y_+, λ_+) and (Y_-, λ_-) be closed, contact 3-manifolds. A *symplectic cobordism* from Y_+ to Y_- is a compact symplectic 4-manifold (X, ω) such that $\partial X = -Y_- \sqcup Y_+$ and $\omega|_{Y_\pm} = d\lambda_\pm$. Given a symplectic cobordism, it is a standard fact that one can always find neighborhoods N_\pm of Y_\pm in X such that (N_+, ω) and (N_-, ω) are symplectomorphic to $((-\epsilon, 0] \times Y_+, d(e^s \lambda_+))$ and $([0, \epsilon) \times Y_-, d(e^s \lambda_-))$ respectively.

We can therefore attach “cylindrical ends” to (X, ω) to obtain a non-compact symplectic manifold \overline{X} called the *symplectic completion* of X . Specifically, define $E_+ := [0, \infty) \times Y_+$ and $E_- := (-\infty, 0] \times Y_-$. Then (\overline{X}, ω) is the symplectic manifold obtained by gluing E_\pm to Y_\pm via the above identifications.

Let X be a symplectic cobordism from Y_+ to Y_- . If $\alpha^+ = \{(\alpha_i^+, m_i^+)\}$ is an orbit set in Y_+ and $\alpha^- = \{(\alpha_j^-, m_j^-)\}$ is an orbit set in Y_- such that $[\alpha^+]$ and $[\alpha^-]$ represent the same class in $H_1(\overline{X})$, define $H_2(\overline{X}, \alpha^+, \alpha_-)$ to be the set of relative homology classes of 2-chains in \overline{X} such that

$$\partial Z = \sum_i m_i^+ \{1\} \times \alpha_i^+ - \sum_j m_j^- \{-1\} \times \alpha_j^-.$$

Here, two 2-chains are equivalent if and only if their difference is the boundary of a 3-chain.

Let τ be a homotopy class of symplectic trivializations τ_i^+ of the restriction of $\xi_+ = \text{Ker}(\lambda_+)$ to α_i^+ and τ_j^- of the restriction of $\xi_- = \text{Ker}(\lambda_-)$ to α_j^- . Let $Z \in H_2(\overline{X}, \alpha^+, \alpha_-)$. Define the ECH index, $I_{ECH}(Z)$ by the formula

$$I_{ECH}(Z) := c_\tau(Z) + Q_\tau(Z) + \mu_\tau(\alpha^+) - \mu_\tau(\alpha^-). \quad (2.3)$$

Here, $c_\tau(Z)$ and $Q_\tau(Z)$ are, respectively, the *relative first Chern class* and *relative intersection pairing* of Z with respect to the trivialization τ , as defined in [5, §4.2].

In the case where $(\overline{X}, \omega) = (\mathbb{R} \times Y, d(e^s \lambda))$, the ECH index induces a relative \mathbb{Z}/p grading on $ECH_*(Y, \lambda, \Gamma)$. This is explained, for example, in [5, §2.8].

3 Seiberg-Witten Floer cohomology

We now review the aspects of Seiberg-Witten Floer cohomology that are relevant to the proof of Theorem 1.1. For more details, see [11].

3.1 Spin^c Structures, The Gauge Group, and The Dirac Operator

We begin by reviewing the basic definitions associated to a spin^c structure. Let Y be a closed oriented Riemannian 3-manifold, and let $\mathfrak{s} = (\mathbb{S}, \rho)$ be a spin^c structure on Y . A *spinor* is a smooth section of \mathbb{S} . A unitary connection \mathbb{A} on \mathbb{S} is called a *spin^c connection* if parallel transport via \mathbb{A} is compatible with Clifford multiplication. The set of spin^c connections is an affine space over the space of imaginary valued 1-forms. Associated to a spin^c structure is the *determinant* line bundle, $\det(\mathbb{S})$. This is the line bundle $\Lambda^2 \mathbb{S}$. If \mathbb{A} is a spin^c connection, we denote by \mathbb{A}^t the induced connection on $\Lambda^2 \mathbb{S}$. A spin^c connection is equivalent to a Hermitian connection on $\Lambda^2 \mathbb{S}$, as shown in [6, §8.2]. Spin^c structures always exist, and the set of isomorphism classes of spin^c structures is an affine space over $H^2(Y, \mathbb{Z})$.

Given a spin^c connection A , define the Dirac operator, $D_{\mathbb{A}}$, to be the composition

$$\Gamma(Y, \mathbb{S}) \xrightarrow{\nabla_A} \Gamma(Y, T^*X \otimes \mathbb{S}) \xrightarrow{\rho} \Gamma(Y, \mathbb{S}).$$

Here, the Clifford multiplication ρ by 1-forms is defined by the isomorphism between vector fields and 1-forms induced by the metric.

Over a closed, oriented, Riemannian 4-manifold X , a spin^c structure \mathfrak{s}_X is again a unitary, complex vector bundle \mathbb{S} , this time of rank 4, together with a Clifford multiplication $\rho : TY \rightarrow \text{Hom}(\mathbb{S}, \mathbb{S})$. Clifford multiplication is characterized by the property that given any $x \in X$, we can find an oriented orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for $T_x X$ and a basis for \mathbb{S}_x such that the linear transformations $\rho(e_1), \rho(e_2), \rho(e_3)$ and $\rho(e_4)$ are given by the matrices

$$\rho(e_0) = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}, \quad \rho(e_i) = \begin{pmatrix} 0 & -\sigma_i^* \\ \sigma_i & 0 \end{pmatrix}, \quad (3.1)$$

where I_2 is the 2-by-2 identity matrix.

Spin^c structures also exist over any 4-manifold, and the set of isomorphism classes of spin^c structures is again an affine space over $H^2(X, \mathbb{Z})$ [11, §1.1].

Clifford multiplication extends to forms by the rule

$$\rho(\alpha \wedge \beta) = \frac{1}{2}(\rho(\alpha)\rho(\beta) + (-1)^{\deg(\alpha)\deg(\beta)}\rho(\beta)\rho(\alpha)),$$

and Clifford multiplication by the volume form induces a decomposition of \mathbb{S} into two orthogonal rank-2 complex vector bundles, S^+ and S^- , where S^+ is defined to be the -1 eigenspace of Clifford multiplication by the volume form. Spin^c connections and spinors for a spin^c structure over a 4-manifold are defined analogously, and we denote by A^t the induced connection on $\Lambda^2 S^+$. As in the three dimensional case, the space of spin^c connections on \mathfrak{s}_X is an affine space over iT^*X .

If \mathfrak{s}_X is a spin^c structure over a 4-manifold, we define the Dirac operator $D_{\mathbb{A}}$ as in the 3-manifold case. In the 4-manifold case, the Dirac operator interchanges sections of S^+ and S^- and hence we have a decomposition $D_{\mathbb{A}} = D_{\mathbb{A}^+} + D_{\mathbb{A}^-}$ where

$$D_{\mathbb{A}^+} : \Gamma(S^+) \rightarrow \Gamma(S^-), \quad D_{\mathbb{A}^-} : \Gamma(S^-) \rightarrow \Gamma(S^+). \quad (3.2)$$

In any dimension, an automorphism of a spin^c structure (\mathbb{S}, ρ) is a bundle isomorphism of \mathbb{S} that is compatible with ρ . This is the same as a map from the underlying manifold into S^1 . We call the set of maps from the underlying manifold to S^1 the *gauge group* and we call elements of this group *gauge transformations*.

If M is a 3-manifold or a 4-manifold and \mathfrak{s} is a spin^c structure over M , denote by $\mathcal{C}(Y, \mathfrak{s})$ the space of pairs (\mathbb{A}, Ψ) such that \mathbb{A} is a spin^c connection and Ψ is a spinor. We call such a pair a *configuration* and call \mathcal{C} the *configuration space*. The gauge group acts on \mathcal{C} by

$$g \cdot (\mathbb{A}, \Psi) := (\mathbb{A} - 2g^{-1}dg, g\Psi).$$

3.2 The Chern-Simons-Dirac functional and the Seiberg-Witten equations

We now present the definition of the three-dimensional Seiberg-Witten equations. Let Y be a closed oriented Riemannian 3-manifold with spin^c structure $\mathfrak{s} = (\mathbb{S}, \rho)$. Fix an exact 2-form μ on Y . The *three-dimensional Seiberg-Witten equations with perturbation* are the equations for a configuration (\mathbb{A}, Ψ) given by

$$\begin{aligned} D_{\mathbb{A}}\Psi &= 0, \\ *F_{\mathbb{A}^t} &= \langle \rho(\cdot)\Psi, \Psi \rangle + i * \mu. \end{aligned} \quad (3.3)$$

Here, $F_{\mathbb{A}^t}$ denotes the curvature of \mathbb{A}^t . Fix a reference spin^c connection \mathbb{A}_0 . Solutions of (3.3) are equivalent to critical points of the *perturbed Chern-Simons-Dirac functional*. This is the map $\mathcal{F} : \mathcal{C}(Y, \mathfrak{s}) \rightarrow \mathbb{R}$ defined by

$$F(A, \varphi) = -\frac{1}{8} \int_Y (\mathbb{A}^t - \mathbb{A}_0^t) \wedge (F_{\mathbb{A}^t} + F_{\mathbb{A}_0^t} - 2i\mu) + \frac{1}{2} \int_Y \langle D_{\mathbb{A}}\varphi, \varphi \rangle d\text{vol}. \quad (3.4)$$

The action of the gauge group on \mathcal{C} induces an action on solutions to (3.3).

3.3 Floer Homology

This section briefly reviews the details of the construction of the Seiberg-Witten Floer cohomology groups. Call a solution to (3.3) *reducible* if $\Psi = 0$ and call it *irreducible* otherwise. The Seiberg-Witten Floer cohomology chain complex $\widehat{CM}^*(Y, \mathfrak{s})$ can be decomposed into submodules

$$\widehat{CM}^*(Y, \mathfrak{s}) = \widehat{CM}_{irr}^*(Y, \mathfrak{s}) \oplus \widehat{CM}_{red}^*(Y, \mathfrak{s}),$$

where \widehat{CM}_{irr}^* is the free $\mathbb{Z}/2$ -module generated by gauge equivalence classes of irreducible solutions to (3.3) after choosing μ generically so that these solutions are cut out transversely and \widehat{CM}_{red}^* is another term involving the reducible solutions. Only the irreducible component of this chain complex is relevant to the construction of the map T from (1.3), so we will not review the definition of \widehat{CM}_{red}^* here.

The part of the chain complex differential ∂ mapping the irreducible component to itself counts gauge equivalence classes of smooth one-parameter families of pairs $(\mathbb{A}(s), \Psi(s))$ that solve the equations

$$\begin{aligned}\frac{\partial}{\partial s}\Psi(s) &= -D_{\mathbb{A}(s)}\Psi(s), \\ \frac{\partial}{\partial s}\mathbb{A}(s) &= -*F_{\mathbb{A}(s)} + \langle cl(\cdot)\Psi, \Psi \rangle + i*\mu, \\ \lim_{s \rightarrow \pm\infty} (\mathbb{A}(s), \Psi(s)) &= (\mathbb{A}_{\pm}, \Psi_{\pm}),\end{aligned}\tag{3.5}$$

where $(\mathbb{A}_{\pm}, \Psi_{\pm})$ are solutions to (3.3). These are equations for the downward gradient flow of the functional (3.4) with respect to the metric on \mathcal{C} induced by the Hermitian inner product on \mathbb{S} and $1/4$ of the L^2 inner product on iT^*Y . Solutions to (3.5) are called *instantons*. If \mathfrak{c}_{\pm} are two irreducible solutions to (3.3), then the coefficient of \mathfrak{c}_{-} in the differential of \mathfrak{c}_{+} is a signed count of gauge equivalence classes of “index one” instantons from \mathfrak{c}_{-} to \mathfrak{c}_{+} , modulo translation in the s coordinate, after making “abstract perturbations” to (3.3) and (3.5) to obtain transversality of the relevant moduli spaces.

“Abstract perturbations” are described in [11, Ch. 11] and play little role in the proof of Theorem 1.1. The “index” is the local expected dimension of the moduli space of instantons modulo gauge equivalence. The index induces a relative \mathbb{Z}/p grading on the chain complex such that the differential increases the grading by 1, see [8, §2.1]. Here, p is equal to the divisibility of $c_1(\mathfrak{s})$ in $H^2(Y, \mathbb{Z})$ mod torsion.

3.4 The Absolute Grading of a Critical Point

We now review the definition of the absolute grading for $\widehat{HM}^*(Y, \mathfrak{s})$. If X is any (possibly non-compact) spin^c 4-manifold, the *four dimensional Seiberg-Witten equations with perturbation* on X for a configuration (A, Ψ) is the system

$$\begin{aligned}\frac{1}{2}\rho(F_{A^t}^+) + \mathfrak{p}(A, \Psi) - (\Psi\Psi^*)_0 &= 0 \\ D_A^+\Psi &= 0.\end{aligned}\tag{3.6}$$

Here, $F_{A^t}^+$ denotes the self-dual part of the curvature 2-form, $(\Psi\Psi^*)_0$ denotes the traceless component of $\Psi\Psi^*$, and $\mathfrak{p}(A, \psi)$ denotes a gauge invariant perturbation term, see [11, §24.1]. When $X = \mathbb{R} \times Y$ and \mathfrak{s} is a certain spin^c structure The action of the gauge group on \mathcal{C} induces an action on solutions of (3.6).

Let \mathfrak{c} be an irreducible “nondegenerate” (see [11, Def. 12.1.1] for the definition of nondegenerate) solution to (3.3) and let X be any compact, connected, oriented Riemannian 4-manifold with oriented boundary Y extending the spin^c structure \mathfrak{s} via a spin^c structure \mathfrak{s}_X . Assume that the Riemannian metric on X is such that X contains an isometric copy of $I \times Y$ for some interval $I = (-C, 0]$ with ∂X identified with $\{0\} \times Y$. We can therefore *attach a cylindrical end to X* i.e. glue in a copy of the cylinder $Z = [0, \infty) \times Y$ to X to get a non-compact 4-manifold \overline{X} with spin^c structure $\mathfrak{s}_{\overline{X}}$ extending the spin^c structure on \overline{X} via a translation invariant spin^c structure on Z .

Denote the moduli space of gauge equivalence classes of configurations for the spin^c structure $\mathfrak{s}_{\overline{X}}$ that are “asymptotic” to \mathfrak{c} on the cylindrical end of \overline{X} by

$\mathcal{B}(\overline{X}, \mathfrak{s}_{\overline{X}}, \mathfrak{c})$ and denote the gauge equivalence classes of solutions to (3.6) that are asymptotic to \mathfrak{c} on the cylindrical end of \overline{X} by $M(\overline{X}, \mathfrak{s}_{\overline{X}}, \mathfrak{c})$. Here, the perturbation term to (3.6) is constructed from the perturbation to (3.3), see [11, §24.1]. Denote by $\mathcal{B}(\overline{X}, \mathfrak{c})$ and by $M(\overline{X}, \mathfrak{c})$ the union of $\mathcal{B}(\overline{X}, \mathfrak{s}_{\overline{X}}, \mathfrak{c})$ and $M(\overline{X}, \mathfrak{s}_{\overline{X}}, \mathfrak{c})$ respectively over all spin^c structures $\mathfrak{s}_{\overline{X}}$ on \overline{X} extending \mathfrak{s} . See [11, §13.1] for the definition of asymptotic.

In general, the space $M(\overline{X}, \mathfrak{c})$ can contain multiple connected components. These are parametrized by $\pi_0(\mathcal{B}(\overline{X}, \mathfrak{c}))$, which is an affine space over $H^2(X, Y, \mathbb{Z})$. Let z be an element of $\pi_0(\mathcal{B}(\overline{X}, \mathfrak{c}))$. We now define an integer $gr_z(X, \mathfrak{c})$, which is the expected dimension of the component of $M(\overline{X}, \mathfrak{c})$ corresponding to z . If (A, Ψ) is any element of $\mathcal{B}(\overline{X}, \mathfrak{c})$, define the operator

$$D_{A, \Psi}^{\overline{X}} : L_1^2(iT^*\overline{X}) \oplus L_1^2(S^+) \rightarrow L^2(i\mathbb{R}) \oplus L^2(isu(S^+)) \oplus L^2(S^-)$$

by

$$D_{A, \Psi}^{\overline{X}}(a, \varphi) = (-d^*a + i\text{Im}(\Psi^*\varphi), \frac{1}{2}\rho(d^+a) - (\Psi\varphi^* + \varphi\Psi^*)_0, D_A^+\varphi + \rho(a)\Psi), \quad (3.7)$$

where $L_1^2(iT^*\overline{X})$, $L_1^2(S^+)$, $L^2(i\mathbb{R})$, $L^2(isu(S^+))$, and $L^2(S^-)$ denote Sobolev completions of the space of compactly supported smooth sections of these bundles over \overline{X} , see [11, §13], and d^+a denotes the self-dual component of da . This is the linearization of the unperturbed 4-dimensional Seiberg-Witten equations with a gauge fixing term. As explained in [13, §3.d] and [15, Lem. 2.4], when \mathfrak{c} is irreducible and nondegenerate, the operator $D_{A, \Psi}^{\overline{X}}$ is Fredholm. The integer $gr_z(X, \mathfrak{c})$ is by definition the index of $D_{A, \Psi}^{\overline{X}}$ for (A, Ψ) a lift of the gauge equivalence class of an element in the component of $\mathcal{B}(\overline{X}, \mathfrak{c})$ corresponding to z . As explained in [11, §24], $gr_z(X, \mathfrak{c})$ can be defined for reducible solutions as well.

If φ_0 is any section of $\mathbb{S}^+|_Y$, denote by $e(\mathbb{S}^+, \varphi_0)$ the *relative Euler class* of $\mathbb{S}^+|_Y$ relative to φ_0 . To define the absolute grading, choose a nowhere-zero section ϕ_0 of $\mathbb{S}^+|_Y$ such that $e(\mathbb{S}^+, \phi_0)[X, \partial X] = gr_z(X; \mathfrak{c})$. The pair $(\mathbb{S}^+|_Y, \varphi_0)$ is a spin^c structure on Y equipped with a non-zero section, so we can apply the following basic lemma [11, Lem. 28.1.1]:

Lemma 3.1. *On an oriented Riemannian 3-manifold Y , there is a one-to-one correspondence between oriented 2-plane fields ξ and isomorphism classes of pairs (\mathfrak{s}, φ) consisting of a spin^c structure and a unit-length spinor φ .*

By [11, Prop. 28.2.2], the isomorphism class of (\mathbb{S}, φ_0) depends only on Y, \mathfrak{s} , and \mathfrak{c} , and so the bijection of Lemma 3.1 induces a well-defined grading by homotopy classes of 2-plane fields.

4 Taubes' Isomorphism

This section very briefly summarizes Taubes' isomorphism between embedded contact homology and Seiberg-Witten Floer cohomology. For more details, see [12].

4.1 Taubes' Equations

Let (Y, λ) be a contact manifold. Fix a metric on Y such that $*d\lambda = 2\lambda$ and $|\lambda| = 1$. Let \mathbb{S} be the spin bundle for the spin^c structure $\mathfrak{s}_\xi + \text{PD}(\Gamma)$. Clifford multiplication by λ gives a decomposition

$$\mathbb{S} = E \oplus (E \otimes \xi)$$

where E and $E \otimes \xi$ are, respectively, the $+i$ and $-i$ eigenspaces of Clifford multiplication by λ . Here ξ is regarded as a complex line bundle.

Connections on $\det \mathbb{S}$ can therefore be written as $A_0 + 2A$ where A_0 is a certain fixed connection on ξ , as reviewed in [16, §2.a], and A is a connection on E . We can therefore regard a connection on E as a connection on $\det \mathbb{S}$. With this in mind, consider the system of equations for a connection A on E and a spinor ψ given by

$$\begin{aligned} *F_A &= r(\langle \rho(\cdot)\psi, \psi \rangle - i\lambda) + i(*d\mu + \bar{\omega}) \\ D_A\psi &= 0. \end{aligned} \tag{4.1}$$

Here, $\bar{\omega}$ denotes the harmonic 1-form such that $*\frac{\bar{\omega}}{\pi}$ represents the image of $c_1(\xi)$ in $H^2(Y; \mathbb{R})$, r is a positive real number, and μ is a suitably generic coclosed 1-form that is L^2 -orthogonal to the space of harmonic 1-forms and that has “P-norm” less than 1. The P-norm controls the derivatives of μ to all orders, see [8, §2.2]. This is a special case of (3.3) where we have also rescaled the spinor by \sqrt{r} .

If μ is generic, then all of the solutions to (4.1) are “nondegenerate” i.e. they are cut out transversely. One can also make additional small perturbations to the equations so that the moduli spaces needed to define the chain complex differential are all cut out transversely. Moreover, in any fixed grading, if r is sufficiently large, these additional perturbations can be chosen such that only irreducible solutions to this perturbed version of (4.1) contribute to the Seiberg-Witten cohomology chain complex in that grading, see [16, Prop. 3.5]. By [8, §2.1], these perturbations can be chosen to vanish to any given order on the irreducible solutions to (4.1), so that the irreducible solutions to (4.1) and the solutions to this perturbed version of (4.1) are the same.

Nondegeneracy is a condition concerning the kernel of a certain linear operator parametrized by the solution, see [16, §3.a]. For our purposes, the key fact about nondegeneracy is that if (A_1, ψ_1) and (A_2, ψ_2) are nondegenerate solutions to (4.1) and γ is a configuration on $\mathbb{R} \times Y$ that is asymptotic to (A_1, ψ_1) and (A_2, ψ_2) , then the linearization of the four-dimensional Seiberg-Witten equations (with an appropriate gauge fixing term) at γ is Fredholm. This is explained in different notation in [13, §3.d].

4.2 Filtered ECH

The basic idea behind the isomorphism (1.3) is that as r gets very large, the zero set of the E component of the spinor for solutions of (4.1) converges (as a current) to an ECH chain complex generator, and the “symplectic action” of this chain complex generator is very close to 2π times the energy (as defined in §4.3) of the solution.

To state this precisely, if $\alpha = \{(\alpha_i, m_i)\}$ is a generator of the ECH chain complex, define the *symplectic action* of α by:

$$\mathcal{A}(\alpha) := \sum_i m_i \int_{\alpha_i} \lambda.$$

Because of the conditions on J , the ECH differential decreases the symplectic action. Hence, for any real number L , we can define *filtered ECH*

$$ECH^L(Y, \lambda, \Gamma)$$

to be the homology of the subcomplex of the ECH chain complex spanned by generators with action strictly less than L .

A priori, $ECH^L(Y, \lambda, \Gamma)$ could depend on the choice of generic J required to define the chain complex differential, but in fact [8, Thm. 1.3] it does not.

4.3 The Energy Filtration

Given a configuration (A, ψ) , define the *energy*

$$E(A) := i \int_Y \lambda \wedge F_A. \quad (4.2)$$

and define $\widehat{CM}_L^*(Y, \mathfrak{s}, \lambda, r)$ to be the submodule of \widehat{CM}_{irr}^* generated by irreducible solutions (A, φ) to (3.3) (perturbed as in §4.1) with energy less than $2\pi L$. If r is sufficiently large, and λ has no orbit set of action exactly L , then one can show [8, Lem. 2.3] that all of the solutions to (4.1) with energy less than $2\pi L$ are irreducible and the chain complex differential for $\widehat{CM}^*(Y, \mathfrak{s}, \lambda, r)$ maps $\widehat{CM}_L^*(Y, \mathfrak{s}, \lambda, r)$ to itself. Denote the homology of this subcomplex with respect to this differential by $\widehat{HM}_L^*(Y, \mathfrak{s}, \lambda, r)$. If r is larger than some (λ, J) -dependent constant, then the homology of this complex does not depend on r , see [8, Cor. 25].

4.4 Taubes' Proof

The key fact ([8, Prop. 3.1]) needed for the proof of (1.3) is that if r is sufficiently large and λ is “L-flat”, then for any $\Gamma \in H_1(Y)$, there is a canonical bijection between the set of generators of $\widehat{CM}_L^{-*}(Y, \mathfrak{s}_\xi + \text{PD}(\Gamma); \lambda, r)$ and the set of admissible orbit sets in the homology class Γ of length less than L . This induces an isomorphism of chain complexes

$$ECC_*^L(Y, \lambda, \Gamma) \xrightarrow{\sim} \widehat{CM}_L^{-*}(Y, \mathfrak{s}_\xi + \text{PD}(\Gamma); \lambda, r). \quad (4.3)$$

Roughly speaking, the map in (4.3) is given by constructing an approximate solution to (4.1) for large r from an ECH chain complex generator by using the *vortex equations*, see [13]. One then uses perturbation theory to get an actual solution to (4.1). The isomorphism T from (1.3) is induced by this map, see [8, §3]. The “L-flat” condition is a technical condition concerning the form of the contact structure in tubular neighborhoods of Reeb orbits with action less than L , see [8, §3.1]. A contact form can always be approximated by an L -flat one, see [8, §3.1].

5 Proof of Main Theorem

5.1 The Seiberg-Witten Index In A Symplectic Cobordism

Let (X, ω) be a symplectic cobordism between two contact 3-manifolds (Y_1, λ_1) and (Y_2, λ_2) as in §2.3, and denote by \overline{X} the symplectic completion of X . Let J be an admissible almost complex structure on \overline{X} , and let g be the Riemannian metric induced by ω and J . Let α_1 be an orbit set on Y_1 and let α_2 be an orbit set on Y_2 . Assume that the contact forms λ_1 and λ_2 are “L-flat”, where L is some constant greater than the symplectic action of either α_1 or α_2 .

Recall that the canonical isomorphism (4.3) is induced from a canonical bijection between the set of generators of $\widehat{CM}_L^{-*}(Y, \mathfrak{s}_\xi + \text{PD}(\Gamma); \lambda, r)$ and the set of admissible orbit sets in the homology class Γ of length less than L . Denote by c_{α_1} and c_{α_2} the elements corresponding to α_1 and α_2 respectively under this bijection. By [15, §2.a], if r is sufficiently large, then c_{α_1} and c_{α_2} are both nondegenerate and belong to the irreducible component of the chain complex \widehat{CM}^* . Regard c_{α_1} and c_{α_2} as gauge equivalence classes of solutions of (4.1).

Let \mathfrak{s}_{Y_1} and \mathfrak{s}_{Y_2} denote the spin^c structures on Y_1 and Y_2 corresponding to c_{α_1} and c_{α_2} respectively. Then $c_{\alpha_1}, c_{\alpha_2}, \mathfrak{s}_{Y_1}$, and \mathfrak{s}_{Y_2} induce a spin^c structure \mathfrak{s}_Y and configuration \mathfrak{c} on $Y = Y_1 \cup -Y_2$.

Recall the space $\mathcal{B}(\overline{X}, \mathfrak{c})$ from §3.4. Let (\mathbb{A}, Ψ) be an element of $\mathcal{B}(\overline{X}, \mathfrak{c})$. Then (\mathbb{A}, Ψ) determines a spin^c structure $\mathfrak{s}_{(\mathbb{A}, \Psi)}$ over \overline{X} . As before, denote by S^+ the -1 eigenspace of Clifford multiplication by the volume form on the spin^c structure $\mathfrak{s}_{(\mathbb{A}, \Psi)}$.

Since \overline{X} is symplectic, we can write $S^+ = E \oplus (E \otimes K^{-1})$, where K^{-1} denotes the inverse of the canonical bundle and E and $E \otimes K^{-1}$ are, respectively, the $-2i$ and $+2i$ eigenspaces of Clifford multiplication by the symplectic form. This is reviewed, for example, in [10, §4.2]. We can then write the spinor $\Psi = (\alpha, \beta)$ according to this decomposition, where (\mathbb{A}, Ψ) now denotes a specific lift of its gauge equivalence class. Assume that (\mathbb{A}, Ψ) is such that α intersects the zero section transversally. Hence, $\alpha^{-1}(0)$ is an embedded (real) surface. Denote this surface by $C_{\mathbb{A}, \Psi}$. Recall the ECH index I_{ECH} of surfaces in cobordisms from §2.3. We have the following theorem:

Theorem 5.1. *Let $z \in \pi_0(\mathcal{B}(\overline{X}, \mathfrak{c}))$. Represent z by a configuration (\mathbb{A}, Ψ) over \overline{X} . The integer $gr_z(X, \mathfrak{c})$ is equal to $I_{ECH}(C_{\mathbb{A}, \Psi})$.*

Proof. Step 1. If $C_{\mathbb{A}, \Psi}$ is J -holomorphic, then one can use ideas of Taubes to compute the index of $D_{\mathbb{A}, \Psi}^{\overline{X}}$. This is explained in step 3. We can not guarantee that $C_{\mathbb{A}, \Psi}$ is J -holomorphic, however, so we must modify the manifold X to produce a new manifold \tilde{X} in which $C_{\mathbb{A}, \Psi}$ is J -holomorphic. We will also want \tilde{X} to admit a nonvanishing self-dual 2-form that is compatible with the almost complex structure on \tilde{X} . To do this, we mimic Taubes’ argument from [15, §2b].

First, by deforming the configuration (\mathbb{A}, Ψ) within the space $\mathcal{B}(\overline{X}, \mathfrak{c})$, we can assume that $C_{\mathbb{A}, \Psi}$ has no compact components and has ends of the special form described in [15, §2b.1]. We can then copy the argument in [15, §2b.2] to produce a pair (J_C, ω_C) , where J_C is an almost complex struture on a neighborhood of $C_{\mathbb{A}, \Psi}$ in \overline{X} such that $C_{\mathbb{A}, \Psi}$ is J_C -holomorphic and ω_C is a self-dual 2-form on \overline{X} whose

restriction to $C_{\mathbb{A},\Psi}$ is compatible with J_C in the sense of [15, §2b. 2] and compatible with λ_1 and λ_2 in the sense of [15, §2b.2]. We require that the pair (J_C, ω_C) satisfy several additional technical conditions, as explained in [15, §2b. 2].

Denote by Z the zero locus of the 2-form ω_C . We can assume that Z has the description given in [15, §2b.2]. In particular, we can assume that Z consists of a finite number of embedded circles which are disjoint from $C_{\mathbb{A},\Psi}$. Let T denote a tubular neighborhood of Z that is disjoint from $C_{\mathbb{A},\Psi}$. Copy the argument in [15, §2b.3] to modify the manifold \overline{X} and the metric on \overline{X} in T to obtain a new Riemannian manifold \tilde{X} such that ω_C extends to a nonvanishing self-dual 2-form $\omega_{\tilde{X}}$ on \tilde{X} . The manifold \tilde{X} is obtained from \overline{X} by cutting out a finite number of copies of $S^1 \times D^3$ and then gluing in the same number of $D^2 \times S^2$. The details of this construction are described in [15, §2b.3].

By the argument in [15, §2b.4], the spin^c structure on $\overline{X} - T$ extends to a spin^c structure $\mathfrak{s}_{\tilde{X}}$ on \tilde{X} such that $S_{\tilde{X}}^+$ splits as $E \oplus E\tilde{K}^{-1}$ with respect to Clifford multiplication by ω_X , where \tilde{K}^{-1} denotes the inverse of the canonical bundle on \tilde{X} . By [15, §2b.4], the equation $\tilde{K} = L^2 K$ holds on $\overline{X} - T$, with L as described in [15, §2b.4]. In particular, because of this description for L , we can choose t_1, t_2 such that $Y_1 \times \{t_1\}$ and $Y_2 \times \{t_2\}$ are both in $\overline{X} - T$ and the restriction of L to $Y_1 \times [t_1, \infty)$ and $Y_2 \times (-\infty, t_2]$ is canonically isomorphic to the trivial bundle.

Step 3. To simplify the notation, denote by C the curve $C_{\mathbb{A},\Psi}$ and denote by \tilde{C} the curve C regarded as a curve in \tilde{X} . Given the conditions satisfied by $\tilde{X}, C, J_C, \omega_X$, and $S_{\tilde{X}}^+$ described above, we can follow the procedure from [14, §2.6] to construct an irreducible configuration $(\mathbb{A}_{\tilde{C}}, \Psi_{\tilde{C}})$ on $(\tilde{X}, \mathfrak{s}_{\tilde{X}})$ with large $|s|$ limit gauge equivalent to \mathfrak{c} .

A formula for the index of $D_{\mathbb{A}_{\tilde{C}}, \Psi_{\tilde{C}}}^{\tilde{X}}$ appears in [15, §2c.1]. Specifically, we have

Lemma 5.2. *The index of $D_{\mathbb{A}_{\tilde{C}}, \Psi_{\tilde{C}}}^{\tilde{X}}$ is equal to $I_{ECH}(C) - 2k_L(C)$.*

Proof. First, observe that the index of $D_{\mathbb{A}_{\tilde{C}}, \Psi_{\tilde{C}}}^{\tilde{X}}$ is the same as the index of the operator defined by [15, Eqn. 2.62]. Now copy the argument in [15, §2c.1]. In this section, Taubes is dealing with a manifold \tilde{X} which arises by removing finitely many $S^1 \times D^3$ from the symplectization of a contact 3-manifold Y and gluing in a corresponding number of $D^2 \times S^2$, but this argument generalizes to our case i.e. the case where the symplectization has been replaced by an arbitrary symplectic cobordism with cylindrical ends. \square

In the statement of this lemma, $I_{ECH}(C)$ denotes the ECH index of the relative homology class of C and $k_L(C)$ denotes the relative first Chern class of L evaluated on C , relative to the section 1 on $Y_1 \times \{t_1\}$ and $Y_2 \times \{t_2\}$.

Step 4. We now need to compare the index of $D_{\mathbb{A}_{\tilde{C}}, \Psi_{\tilde{C}}}^{\tilde{X}}$ to the index of $D_{\mathbb{A}, \Psi}^{\overline{X}}$.

Denote the component of \overline{X} bounded by $Y_1 \times \{t_1\}$ and $Y_2 \times \{t_2\}$ by M and denote the corresponding component of \tilde{X} by \tilde{M} . Glue M to \tilde{M} (reversing the orientation on \tilde{M}) along their common boundary to obtain a closed spin^c 4-manifold (S, \mathfrak{s}_S) . Let (\mathbb{A}_0, Ψ_0) be a configuration on \overline{X} that agrees with the configuration $(\mathbb{A}_{\tilde{C}}, \Psi_{\tilde{C}})$ on the complement of M . Because of the definition of $(\mathbb{A}_{\tilde{C}}, \Psi_{\tilde{C}})$, the configurations (\mathbb{A}_0, Ψ_0) and (\mathbb{A}, Ψ) are in the same component of $\mathcal{B}(\overline{X}, \mathfrak{c})$. We therefore have

$$\text{ind}(D_{\mathbb{A}_0, \Psi_0}^{\overline{X}}) = \text{ind}(D_{\mathbb{A}, \Psi}^{\overline{X}}), \quad (5.1)$$

where ind denotes the index of the operator in parentheses.

Now glue the configuration (\mathbb{A}, Ψ) and (\mathbb{A}_0, Ψ_0) to get a configuration (\mathbb{A}_S, Ψ_S) on S . To simplify the notation, let $D_X, D_{\tilde{X}}, D_S$ denote the operators $D_{\mathbb{A}_0, \Psi_0}^X, D_{\mathbb{A}_C, \Psi_C}^{\tilde{X}}$ and $D_{\mathbb{A}_S, \Psi_S}^S$ respectively. Concerning these operators, we have

Lemma 5.3.

$$\text{ind}(D_X) = \text{ind}(D_{\tilde{X}}) + \text{ind}(D_S). \quad (5.2)$$

Proof. This follows from the standard fact that the index of these operators is additive under gluing. \square

By [11, Thm. 1.4.1], we have

$$\text{ind}(D_S) = \frac{1}{4}(c_1(S^+)^2[S] - 2\chi(S) - 3\sigma(S)), \quad (5.3)$$

where σ denotes the signature of S . We also have the following [11, Lem. 28.2.3]

Lemma 5.4. *For a closed 4-manifold S with spin^c structure \mathfrak{s}_S , we have:*

$$(c_2(S^+) - \frac{1}{4}c_1(S^+)^2)[S] = -\frac{1}{4}(2\chi(S) + 3\sigma(S)). \quad (5.4)$$

Combining these two equations gives

$$\text{ind}(D_S) = c_2(S^+)[S]. \quad (5.5)$$

We therefore have

$$\begin{aligned} \text{ind}(D_S) &= 2(c_1(E) \cup c_1(L))[M] \\ &= 2k_L(C). \end{aligned} \quad (5.6)$$

The result now follows by combining Lemma 5.2, Lemma 5.3, (5.6) and (5.1). \square

5.2 A Concave Symplectic Filling

Let $\Gamma \in H_1(Y)$ and fix an orbit set $\alpha \in ECC(Y, \lambda, \Gamma)$. Recall from [1, Thm. 2.5] that any smooth knot can be C^0 approximated by a Legendrian knot. Thus, we can choose a Legendrian knot L which represents the class Γ .

Recall now the concept of Legendrian surgery. This is reviewed, for example, in [2]. Recall also from [2, Prop. 2.1] that if L is a Legendrian knot in a contact 3-manifold (Y, ξ) and (Y', ξ') is another contact 3-manifold obtained from (Y, ξ) by a Legendrian surgery along L then there exists a symplectic cobordism from (Y, λ) to (Y', λ') obtained by attaching a 2-handle along a tubular neighborhood of L . Here, λ and λ' are any contact forms for ξ and ξ' respectively.

Define a *concave symplectic filling* of (Y, ξ) to be a symplectic cobordism from (Y, ξ) to the empty set. Concerning concave symplectic fillings, Etnyre and Honda prove [2, Thm. 1.3] that any contact 3-manifold has infinitely many concave symplectic fillings.

Given $\alpha \in ECC(Y, \lambda, \Gamma)$, we can therefore combine these results to define a manifold X_α as follows: first perform Legendrian surgery on Y along L to obtain another contact 3-manifold and then compose the resulting symplectic cobordism with a concave symplectic filling to obtain a concave symplectic filling of (Y, λ) . This is the manifold X_α . Note that the image of Γ in X_α is equal to 0.

5.3 Completing the proof

Proof of Theorem 1.1. Let $\alpha = \{(\alpha_i, m_i)\}$ be a generator of $ECC(Y, \lambda, \Gamma)$ and assume that the symplectic action of α is less than L . Let X_α be the symplectic manifold defined in the previous section. Denote by c_α the element corresponding to α under the canonical bijection between the set of generators of $\widehat{CM}_L^{-*}(Y, \mathfrak{s}_\xi + PD(\Gamma); \lambda, r)$ and the set of admissible orbit sets in the homology class Γ of length less than L . Let b_i be a braid with m_i strands around α_i , and let b be the union of the b_i over all i . Let τ_i be a trivialization of ξ over each α_i and denote by τ the set of all τ_i .

By Lemma 3.1, the 2-plane field $\tilde{\xi} = P_\tau(b)$ determines a pair (S^+, φ) , where S^+ is the spin bundle for a spin^c structure \mathfrak{s} on Y and φ is a nowhere zero section of S^+ . Let \overline{X}_α denote the manifold X_α with cylindrical ends attached. Because $\Gamma = 0$ in $H_1(X_\alpha)$, we can extend \mathfrak{s} to a spin^c structure on \overline{X}_α , which we will also denote by \mathfrak{s} .

Let Ψ be a section of S^+ extending φ and transverse to the zero section, and write $S^+ = E \oplus E \otimes K^{-1}$ over X_α . Write $\Psi = (\gamma, \tilde{\gamma})$ with respect to this decomposition. The zero set of the γ component of E defines an embedded real surface in X_α , denoted C_α . By Theorem 5.1 we can choose $z \in \pi_0(\mathcal{B}(\overline{X}_\alpha, \mathfrak{c}_\alpha))$ such that

$$I_{ECH}(C_\alpha) = gr_z(X_\alpha, \mathfrak{c}_\alpha). \quad (5.7)$$

As explained in §3.4, the Seiberg-Witten grading of \mathfrak{c}_α is the homotopy class of two-plane fields corresponding to (S^+, φ_0) , where φ_0 is a section of S^+ satisfying

$$e(S^+, \varphi_0)[X_\alpha, \partial X_\alpha] = gr_z(X_\alpha; \mathfrak{c}_\alpha). \quad (5.8)$$

To simplify the notation, for a section φ of S^+ , denote by $\tilde{e}(S^+, \varphi)$ the number $e(S^+, \varphi)[X_\alpha, \partial X_\alpha]$.

Denote by $I_{SW}(\mathfrak{c}_\alpha)$ the Seiberg-Witten grading of the generator \mathfrak{c}_α . It follows from (5.8) and the properties of the relative Euler class that

$$I_{SW}(\mathfrak{c}_\alpha) = (S^+, \varphi) - \tilde{e}(S^+, \varphi) + gr_z(X_\alpha; \mathfrak{c}_\alpha), \quad (5.9)$$

with φ defined as above.

By (5.7), we therefore have

$$I_{SW}(\mathfrak{c}_\alpha) = (S^+, \varphi) - \tilde{e}(S^+, \varphi) + I_{ECH}(C_\alpha). \quad (5.10)$$

A zero of Ψ is precisely a zero of $\tilde{\gamma}$ over C_α . Moreover, we know that

$$C^\infty(E|_{C_\alpha} \otimes K^{-1}|_{C_\alpha}) = C^\infty(E|_{C_\alpha}) \otimes_{C^\infty(C_\alpha, \mathbb{C})} C^\infty(K^{-1}|_{C_\alpha}).$$

The map $d\gamma$ induces an isomorphism between the normal bundle to C_α and the restriction of the bundle E to C_α . Moreover, as shown below, $\partial C_\alpha = b$. Since

projection induces a canonical isomorphism between $\xi|_{\partial C_\alpha}$ and the normal bundle to C_α restricted to ∂C_α , the trivialization τ therefore induces a trivialization of E over ∂C_α . Define $c_1(E|_{C_\alpha}, \tau)$ and $c_1(K^{-1}|_{C_\alpha}, \tau)$ to be a signed count of the zeros of a generic section of $E|_{C_\alpha}$ and $K^{-1}|_{C_\alpha}$ respectively extending a non-zero section over ∂C_α that has winding number 0 with respect to τ .

Deforming $\tilde{\gamma}$ if necessary to be sufficiently generic, we can write $\tilde{\gamma} = e \otimes k$, where e is a section of $E|_{C_\alpha}$ and k is a section of $K^{-1}|_{C_\alpha}$. Moreover, we have

Proposition 5.5. $\partial C_\alpha = b$ and $\tilde{\gamma} = e \otimes k$, where $e|_{\partial C_\alpha}$ and $k|_{\partial C_\alpha}$ have winding number 0 with respect to τ .

Proof. Let U_j be a tubular neighborhood of one of the components for one of the b_i ; assume that U_j is small enough so that U_j does not contain any other components of any of the b_i . Recall from §2.2 that there are coordinates $\varphi : \mathbb{R} \oplus \mathbb{C} \xrightarrow{\cong} TU_i$ such that the Reeb vector field is always given by $\langle 1, 0, 0 \rangle$ and ξ is given by $\{0\} \oplus \mathbb{C}$. Assume as well that the coordinates are such that the Riemannian metric is given by the standard Euclidean metric. Let (t, r, θ) be these coordinates.

Over each U_j , define a vector field P_j in these coordinates by

$$P_j(t, re^{i\theta}) = (-\cos(\pi r), \sin(\pi r)\cos(\theta), -\sin(\pi r)\sin(\theta)). \quad (5.11)$$

Extend the P_j by the Reeb vector field to a vector field P on Y . Because the P_j satisfy the conditions described in §2.2, the vector field P is equivalent to $P_\tau(b)$ under the correspondence between vector fields and 2-plane fields.

Over each U_j , $S^+ = \mathbb{C} \oplus \tilde{\xi}$, where $\tilde{\xi}$ is orthogonal to P_j and $\varphi = (1, 0)$. Recall from [11, Lem. 28.1.1] that the Clifford multiplication, ρ , is such that \mathbb{C} is the $+i$ eigenspace of Clifford multiplication by P_j and $\tilde{\xi}$ is the $-i$ eigenspace. Recall as well that for any vector v orthogonal to P_j , $\rho(v)(\varphi) = (0, v)$. Define

$$\tilde{P}_j(t, r, \theta) = (\sin(\pi r), \cos(\pi r)\cos(\theta), -\cos(\pi r)\sin(\theta))$$

We have that \tilde{P}_j and P_j are orthogonal. Moreover

$$\langle 1, 0, 0 \rangle = -\cos(\pi r)P_j + \sin(\pi r)\tilde{P}_j.$$

Hence, in the frame φ and \tilde{P}_j for S^+ , Clifford multiplication by the Reeb vector field is given by

$$\rho(R) = \begin{pmatrix} -i\cos(\pi r) & -\sin(\pi r) \\ \sin(\pi r) & i\cos(\pi r) \end{pmatrix}.$$

In particular, Clifford multiplication by R does not depend on t and the component of φ in the $+i$ eigenspace of $\rho(R)$ vanishes precisely at $r = 0$. The result now follows. \square

Combining (5.10) and Proposition 5.5 gives

$$I_{SW}(\mathfrak{c}_\alpha) = (S^+, \varphi) - (c_1(E|_{C_\alpha}, \tau) + c_1(K^{-1}|_{C_\alpha}, \tau)) + I_{ECH}(C_\alpha). \quad (5.12)$$

From the definitions of I_{ECH} and $c_\tau(Z)$, we can rewrite this equation as

$$I_{SW}(\mathfrak{c}_\alpha) = (S^+, \varphi) + Q_\tau(C_\alpha) - c_1(N, \tau) - \mu_\tau(\alpha). \quad (5.13)$$

Copying the argument in [4, Prop. 3.1] gives

$$c_1(N, \tau) = -\omega_\tau(b) + Q_\tau(C_\alpha). \quad (5.14)$$

Hence, combining these equations, we get that

$$I_{SW}(\mathfrak{c}_\alpha) = (S^+, \varphi) - (\mu_\tau(\alpha) - \omega_\tau(b)). \quad (5.15)$$

By definition, $(S^+, \varphi) = P_\tau(b)$. Taking into account our sign conventions, we therefore have that

$$I_{SW}(\mathfrak{c}_\alpha) = P_\tau(b) + \mu_\tau(\alpha) - \omega_\tau(b). \quad (5.16)$$

By the definition of the absolute grading in ECH from §2.2, the result now follows. \square

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